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CONSOLIDATION OF AN ELASTIC POROUS MULTILAYER MEDIUM UNDER THE PRESSURE OF A CIRCULAR PRESS

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A solution is given for the problem of impressing a circular cylindrical press with a flat base in a packet of an arbitrary number of plane-parallel layers lying on a half-space. The layer and half-space are considered elastic porous bodies saturated with a fluid and possessing different elastic characteristics. Two extreme cases are considered, when the outer surface is completely permeable or impermeable to the interstitial fluid. The condition of total adhesion of the layers is assumed on the boundary planes between the layers.

The general solution, based on [1, 2], of the contact problem for an elastic porous multilayered medium being consolidated is presented in Sect. 1, as expressed in terms of multiple Bromwich and Hankel integrals. The solution of the contact problem is constructed in Sect. 2. Calculation methods used to obtain the numerical solution of the problem are described in Sect. 3 and results of calculating the magnitude of the press settlement and the intensity of its pressure in time are presented in the case of a two-layered half-space with different elastic and filtration characteristics. It is found that if the layer is sufficiently thin, but stiffer than the half-space, then inadmissible tensile stresses originate on the contact area as in the case of the analogous elasticity theory problem [3]. Therefore, the pressure distribution on the contact area between the press and the multilayered half-space being consolidated as a function of its elastc and geometric characteristics can differ qualitatively from the pressure distributions on the contact area between a press and a homogeneous half-space being consolidated.

Theoretical principles for the consolidation of an elastic porous medium have been developed by Biot [4, 5]. The first boundary value problem of the theory of consolidation for a half-space and layer is considered in [6—8] in particular loading cases. The plane contact problem of the theory of consolidation for the impression of a press with a flat base is a homogeneous half-space has been examined in [9, 10].

1. General solution of the problem of consolidation theory for a multilayered medium. An elastic, porous, fluid-saturated, multilayered half-space consisting of N layers at rest on an elastic foundation of infinite extent is considered. Numbers $j = 1, 2, \ldots, N$, from top to bottom are ascribed to each layer and the elastic foundation is considered as the infinitely thick layer $N + 1$. The elastic moduli E_j, Poisson ratios ν_j, consolidation factors $c_j = 2G_j\eta_j k_j$ expressed in terms of the shear moduli $G_j = E_j / 2(1 + \nu_j)$, the permeability coefficients in the Darcy law k_j and $\eta_j = (1 - \nu_j) / (1 - 2\nu_j)$ for each layer $j = 1, 2, \ldots, N + 1$ can take on different and arbitrary admissible values. The condition of total adhesion of the layers is assumed on the boundary planes between the layers. Let us note that the solution of the problem for the case of attached layers is easily modified even for the case of contact of the layers without friction.

The origin of a cylindrical r, z coordinate system is taken on the boundary plane between the layers N and $N+1$. The layers are bounded by parallel planes $z = H_j$ ($j = 0, 1, 2, \ldots, N$) orthogonal to oz-axis in this coordinate system, where $z = H_0$ is the outer boundary plane of the multilayered half-space and the quantity H_0 equals the total thickness of all the layers resting on the elastic foundation. The thicknesses of the layers $H_{j-1} - H_j$ ($j = 1, 2, \ldots, N$) can take on different and arbitrary values (Fig. 1).

The fundamental equations of the theory of consolidation in displacements for the j-th layer in the presence of axial symmetry have the form [6]

$$\left(\Delta - \frac{1}{r^2}\right) u_j - \frac{1}{1-2\nu_j}\frac{\partial e_j}{\partial r} - \frac{1}{G_j}\frac{\partial \sigma_j}{\partial r} = 0 \qquad (1.1)$$

$$\Delta w_j - \frac{1}{1-2\nu_j}\frac{\partial e_j}{\partial z} - \frac{1}{G_j}\frac{\partial \sigma_j}{\partial z} = 0, \quad c_j \Delta e_j = \frac{\partial e_j}{\partial t}$$

where $u_j = u_j(r, z, t)$, $w_j = w_j(r, z, t)$ are the radial and axial displacements, Δ is the Laplace operator, t is the time, σ_j is the fluid pressure in the pores, and e_j is the volume expansion per unit volume.

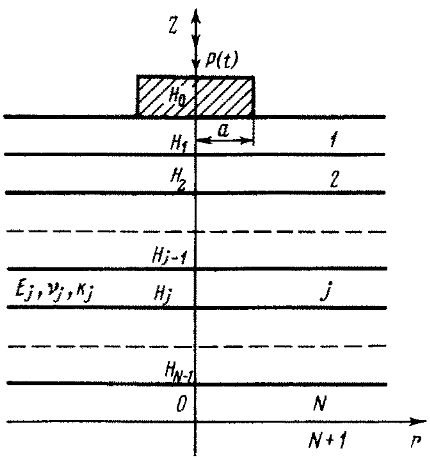

Fig. 1

Let us introduce the dimensionless spatial variables $\rho = r / a$, $y = z / H_0$ and the time $t' = tc / a^2$, where c is the consolidation factor taken as the measurement unit (for example, it can be set equal to c_1 or c_{N+1}), and a is the radius of the circular contact area. The general solution of the contact problem in the dimensionless variables ρ, y, t' depends on the characteristic parameter $\lambda = H_0 / a$, the geometric parameters $y_j = H_j / H_0$ determining the boundaries of the layers $y = y_j$ ($j = 1, 2, \ldots, N$), the elastic parameters $\chi_j = E_j(1 + \nu_{j+1}) / E_{j+1}(1 + \nu_j)$, $\eta_j = (1 - \nu_j) / (1 - 2\nu_j)$ and the filtration parameters $\varkappa_j = k_j / k_{j+1}$, $c_j' = c_j / c$. The primes on the referred time t' and on the parameter c_j' will henceforth be omitted everywhere in the exposition.

It is assumed that there is no friction on the contact area and on the whole external surface $y = 1$ of the multilayered half-space. The intensity of the external normal load distribution $p(\rho, t)$ is represented in terms of a multiple Bromwich-Hankel integral

$$p(\rho, t) = \int_0^\infty \beta \bar{p}(\beta, t) J_0(\rho\beta)\, d\beta \qquad (1.2)$$

$$\bar{p}(\beta, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{s} e^{st} \bar{p}^*(\beta, s)\, ds \quad (\gamma > 0)$$

where the Hankel transform $\bar{p}(\beta, t)$ of $p(\rho, t)$ and the Laplace-Carson transform $\bar{p}^*(\beta, s)$ of $\bar{p}(\beta, t)$ are

$$\bar{p}(\beta, t) = \int_0^\infty \rho p(\rho, t) J_0(\rho\beta)\, d\rho, \quad \bar{p}^*(\beta, s) = s\int_0^\infty \bar{p}(\beta, t)\, e^{-st}\, dt \tag{1.3}$$

If there is no external normal load outside the contact area $0 \leqslant \rho \leqslant 1$, then the function $p(\rho, t)$ is the unknown intensity of the press pressure on the contact area.

The general solution of the contact problem for each layer $j = 1, 2, \ldots, N+1$ of the multilayered half-space being consolidated, which is expressed in terms of the repeated Bromwich-Hankel integral, has been constructed and investigated in [1]. The normal axial stresses $\sigma_{zj}(\rho, y, t)$ and the displacements $w_j(\rho, y, t)$ are represented in an arbitrary layer $j = 1, 2, \ldots, N+1$ by the formulas

$$\sigma_{zj} = \int_0^\infty \beta\Delta_{zj}(y, \beta, t) J_0(\rho\beta)\, d\beta \tag{1.4}$$

$$-\frac{2G_j}{a} w_j = \int_0^\infty \Delta_{wj}(y, \beta, t) J_0(\rho\beta)\, d\beta \tag{1.5}$$

$$\Delta_{mj}(y, \beta, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{p}^*(\beta, s)\, s^{-1} e^{st} \Delta_{mj}^*(y, \beta, s)\, ds \quad (m = z, w) \tag{1.6}$$

Here $\Delta_{mj}^*(y, \beta, s)$ $(m = z, w)$ are expressed in terms of the arbitrary vector function

$$B_j(\beta, s) = [a_j(\beta, s), b_j(\beta, s), c_j(\beta, s), d_j(\beta, s), f_j(\beta, s), g_j(\beta, s)],$$

which corresponds to the j-th layer. The remaining stresses σ_{rj}, $\sigma_{\theta j}$, τ_{rzj}, the radial displacements $-u_j$ and the pressure in the pores σ_j are represented by analogous formulas in terms of $\Delta_{mj}^*(y, \beta, s)$ $(m = r, \theta, rz, u, \sigma)$. It should be kept in mind that the stresses given are considered positive, as is assumed in the theory of solids as contrasted to the theory of elasticity.

It is required of the general solution of the contact problem of the form (1.4), (1.5) that it satisfy conditions on the outer boundary plane in the case of its total permeability (or impermeability) to the interstitial fluid and strain compatibility conditions of adjacent fastened layers on their boundary planes

$$\sigma_{z1} = p(\rho, t), \quad \tau_{rz1} = 0, \quad \sigma_1 = 0 \quad (\text{or} \quad \partial\sigma_1/\partial z = 0) \quad \text{for } y = 1 \tag{1.7}$$

$$\sigma_{zj} = \sigma_{zj+1}, \quad \tau_{rzj} = \tau_{rzj+1}, \quad \sigma_j = \sigma_{j+1} \tag{1.8}$$

$$u_j = u_{j+1}, \quad w_j = w_{j+1}, \quad \varkappa_j \frac{\partial\sigma_j}{\partial z} = \frac{\partial\sigma_{j+1}}{\partial z} \quad \text{for } y = y_j$$

As has been shown in [1], conditions (1.7) and (1.8) reduce to a matrix system of $6N+3$ functional equations to determine the vector B:

$$KB = I \tag{1.9}$$

$$K = \left\| \begin{array}{cccccc} N_1 & & & & & \\ M_1 & R_2 & & 0 & & \\ & M_2 & R_3 & & & \\ & & \ddots & \ddots & & \\ & 0 & & M_{N-1} & R_N & \\ & & & & M_N & P_{N+1} \end{array} \right\|, \quad I = \left\| \begin{array}{c} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \right\|$$

$$B = [B_1(\beta, s),\ B_2(\beta, s), \ldots, B_{N+1}(\beta, s)]$$

$$B_j(\beta, s) = [a_j(\beta, s), b_j(\beta, s), c_j(\beta, s), d_j(\beta, s), f_j(\beta, s), g_j(\beta, s)]$$
$$(j = 1, 2, \ldots, N)$$

$$B_{N+1}(\beta, s) = [a_{N+1}(\beta, s),\ b_{N+1}(\beta, s),\ c_{N+1}(\beta, s)]$$

Here the matrix-squares N_1, M_j, R_j, P_{N+1} have the form ($\delta = 0$ in the case of total permeability of the outer surface, and $\delta = 1$ in the case of total impermeability)

$$N_1 = \left\| \begin{array}{cccccc} 1 & \beta & 1-\beta\lambda & e_1 & \beta e_1^* & -(1+\beta\lambda)e_1 \\ -1 & -q_1 & \beta\lambda & e_1 & q_1 e_1^* & -\beta\lambda e_1 \\ 0 & -\eta_1 s_1 q_1^{\delta} & \beta^{1+\delta} & 0 & -\eta_1 s_1 e_1^* (-q_1)^{\delta} & (-\beta)^{1+\delta} e_1 \end{array} \right\|$$

$$M_j = \left\| \begin{array}{cccccc} e_j & \beta e_j^* & (1-\beta\lambda y_j)e_j & 1 & \beta & -(1+\beta\lambda y_j) \\ -e_j & -q_j e_j^* & \beta\lambda y_j e_j & 1 & q_j & -\beta\lambda y_j \\ 0 & -\eta_j s_j e_j^* & \beta e_j & 0 & -\eta_j s_j & -\beta \\ e_j & \beta e_j^* & -\beta\lambda y_j e_j & 1 & \beta & -\beta\lambda y_j \\ -e_j & -q_j e_j^* & -(1-\beta\lambda y_j)e_j & 1 & q_j & -(1+\beta\lambda y_j) \\ 0 & -\eta_j s_j \varkappa_j q_j e_j^* & \beta^2 \varkappa_j e_j & 0 & \eta_j s_j \varkappa_j q_j & \beta^2 \varkappa_j \end{array} \right\|$$

$$R_j = \left\| \begin{array}{cccccc} -1 & -\beta & -(1-\beta\lambda y_{j-1}) & -e_j & -\beta e_j^* & (1+\beta\lambda y_{j-1})e_j \\ 1 & q_j & -\beta\lambda y_{j-1} & -e_j & -q_j e_j^* & \beta\lambda y_{j-1} e_j \\ 0 & \eta_j s_j & -\beta & 0 & \eta_j s_j e_j^* & \beta e_j \\ -\chi_{j-1} & -\beta\chi_{j-1} & \beta\lambda y_{j-1}\chi_{j-1} & -\chi_{j-1}e_j & -\beta\chi_{j-1}e_j^* & \beta\lambda y_{j-1}\chi_{j-1}e_j \\ \chi_{j-1} & q_j\chi_{j-1} & (1-\beta\lambda y_{j-1})\chi_{j-1} & -\chi_{j-1}e_j & -q_j\chi_{j-1}e_j^* & (1+\beta\lambda y_{j-1})\chi_{j-1}e_j \\ 0 & \eta_j s_j q_j & -\beta^2 & 0 & -\eta_j s_j q_j e_j^* & -\beta^2 e_j \end{array} \right\|$$

$$P_{N+1} = \left\| \begin{array}{ccc} -1 & -\beta & -1 \\ 1 & q_{N+1} & 0 \\ 0 & \eta_{N+1}s_{N+1} & -\beta \\ -\chi_N & -\beta\chi_N & 0 \\ \chi_N & q_{N+1}\chi_N & \chi_N \\ 0 & \eta_{N+1}s_{N+1}q_{N+1} & -\beta^2 \end{array} \right\|$$

$$\left(s_j = s / c_j,\ q_j = \sqrt{s_j + \beta^2},\ e_j = e^{-\beta\lambda(y_{j-1} - y_j)},\ e_j^* = e^{-\lambda q_j(y_{j-1} - y_j)}\right)$$

For fixed s and β the system of functional equations (1.9) becomes an algebraic system from which the column vector $B(\beta, s)$ can be determined.

Taking account of (1.2), (1.4), (1.6) at $y = 1$ we find $\Delta_{z1}^*(1, \beta, s) = 1$ from the first boundary condition (1.7). The function $\Delta_{w1}^*(y, \beta, s)$ is expressed on the outer boundary plane $y = 1$ in terms of the product

$$\Delta_{w1}^* (1, \beta, s) = D (\beta, s) B_1 (\beta, s) \tag{1.10}$$

or the row vector

$$D (\beta, s) = (1,\ q_1,\ 1 - \beta\lambda,\ -e_1,\ -q_1 e_1^*,\ (\beta\lambda + 1) e_1)$$

by the column vector $B_1 (\beta, s)$ determined from the system (1.9). As has been established in [1], the function (1.10) has the following asymptote as $\beta \to \infty$:

$$\Delta_{w1}^* (1, \beta, s) = 2 (1 - \nu_1) + \frac{\gamma s}{\beta^2} + O\left(\frac{s^2}{\beta^2}\right) + o (\beta^m e^{-\alpha\beta}) \tag{1.11}$$
$$(m,\ \gamma,\ \alpha = \mathrm{const} > 0)$$

The repeated Laplace-Carson and Hankel transform $\bar{p}^* (\beta, s)$ of the function $p (\rho, t)$ of the press pressure intensity on the contact area in (1.4) and (1.5), and the analogous formulas for the remaining components of the general solution for the j-th layer are to be determined from the mixed boundary conditions of the contact problem on the outer boundary plane.

2. Construction of the solution of the contact problem. Let us consider the contact problem for the following boundary conditions on the boundary plane $y = 1$ of a multilayered half-space. Axial displacements are given on the contact area between the press with a flat base and the multilayered half-space, while zero axial stresses are given outside the contact area

$$-\frac{2G_1}{a} w_1 (y, \rho, t) |_{y=1} = h (t), \quad 0 \leqslant \rho \leqslant 1 \tag{2.1}$$

$$\sigma_{z1} (y, \rho, t) |_{y=1} = 0, \quad 1 < \rho < \infty \tag{2.2}$$

Moreover, it is assumed that the outer boundary plane is completely permeable or impermeable to the interstitial fluid and is free of tangential stresses. These conditions are given by the second and third equalities in (1.7) and therefore have already been taken into account in constructing the general solution of the problem.

Substituting the representations of the functions $w_1 (y, \rho, t)$ and $\sigma_{z1} (y, \rho, t)$ by means of (1.4), (1.5) into (2.1) and (2.2) and taking account of (1.6) and $\Delta_{z1}^* (1, \beta, s) = 1$, we obtain dual integral equations in $\bar{p}^* (\beta, s)$

$$\int_0^\infty \left\{\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{p}^* (\beta, s) \Delta_{w1}^* (1, \beta, s) e^{st} \frac{ds}{s}\right\} J_0 (\rho\beta) d\beta = h (t), \quad 0 \leqslant \rho \leqslant 1$$

$$\int_0^\infty \left\{\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{p}^* (\beta, s) e^{st} \frac{ds}{s}\right\} \beta J_0 (\rho\beta) d\beta = 0, \quad 1 < \rho < \infty$$

Let us change the order of integration in these equations and let us perform a Laplace-Carson transformation; we consequently obtain

$$\int_0^\infty \bar{p}^* (\beta, s) \Delta_{w1}^* (1, \beta, s) J_0 (\rho\beta) d\beta = h^* (s), \quad 0 \leqslant \rho \leqslant 1 \tag{2.3}$$

$$\int_0^\infty \beta \bar{p}^* (\beta, s) J_0 (\rho\beta) d\beta = 0, \quad 1 < \rho < \infty \tag{2.4}$$

Subtracting its principal term $2(1-\nu_1)$ from the function $\Delta_{w1}^*(1, \beta, s)$ as $\beta \to \infty$, defined by (1.11), we obtain the new function

$$\Delta_w^*(\beta, s) = \Delta_{w1}^*(1, \beta, s) - 2(1-\nu_1) \tag{2.5}$$

By the Hankel inversion formula and taking account of (1.2) and (1.3), it follows from (2.3) that $\bar{p}^*(\beta, s)$ is determined by the integral

$$\bar{p}^*(\beta, s) = \int_0^1 \rho p^*(\rho, s) J_0(\rho\beta)\, d\rho \tag{2.6}$$

where $p^*(\rho, s)$ is the Laplace-Carson transform of the required function of the press pressure intensity $p(\rho, t)$. The function (2.6) converts (2.4) into an identity.

Considering the variable s a parameter, we reduce the integral equation (2.3) with the representations (2.5) and (2.6) taken into account, to a Fredholm integral equation of the second kind for the new unknown function $\varphi^*(x, s)$ by the known Noble [11] or Lebedev-Ufliand [12] methods just as has been done in [3]:

$$\pi(1-\nu_1)\varphi^*(x, s) + \int_0^1 K(x, z, s)\varphi^*(z, s)\, dz = h^*(s), \quad 0 \leqslant x \leqslant 1 \tag{2.7}$$

which has a continuous symmetric kernel

$$K(x, z, s) = \int_0^\infty \Delta_w^*(\beta, s)\cos(x\beta)\cos(z\beta)\, d\beta \tag{2.8}$$

for any s in the square $0 \leqslant x, z \leqslant 1$.

Taking account of (2.5), the asymptotic formula (1.11) evidently assures uniform convergence of the integral (2.8) in the square $0 \leqslant x, z \leqslant 1$ to the function $K(x, z, s)$ for any s.

The Laplace-Carson transform and the repeated Laplace-Carson and Hankel transform of the required function of the press pressure intensity $p(\rho, t)$ are expressed in terms of $\varphi^*(x, s)$ by means of the formulas

$$p^*(\rho, s) = -\frac{1}{\rho}\frac{d}{d\rho}\int_\rho^1 \frac{x\varphi^*(x, s)}{\sqrt{x^2-\rho^2}}\, dx \tag{2.9}$$

$$\bar{p}^*(\beta, s) = \int_0^1 \varphi^*(x, s)\cos(x\beta)\, dx \tag{2.10}$$

Now, let us express the magnitude of the force applied to the press in terms of $\varphi^*(x,s)$

$$P(t) = 2\pi a^2 \int_0^1 \rho p(p, t)\, d\rho \tag{2.11}$$

To do this, we apply a Laplace-Carson transform on (2.11), whereupon we obtain by taking account of (2.9)

$$P^*(s) = 2\pi a^2 \int_0^1 \varphi^*(x, s)\, dx \tag{2.12}$$

Let us represent the solution of the integral equation (2.7) as

$$\varphi^*(x, s) = h^*(s)\varphi_1^*(x, s) \tag{2.13}$$

where the function $\varphi_1^* (x, s)$ satisfies (2.7) for $h^* (s) = 1$. Then (2.12) is rewritten as

$$P^* (s) = 2\pi a^2 h^* (s) \int_0^1 \varphi_1^* (x, s)\, dx \qquad (2.14)$$

In the space of Laplace-Carson transforms, (2.14) establishes the connection between the force $P (t)$ applied to the press and the depth of its submersion $h (t)$.

We rewrite (2.14) in a different form and we go back to the space of originals

$$a^2 h^* (s) = P^* (s) h_1^* (s), \quad h_1^* (s) = \left[2\pi \int_0^1 \varphi_1^* (x, s)\, dx\right]^{-1} \qquad (2.15)$$

$$a^2 h (t) = L^{-1} [P^* (s) h_1^* (s)] \qquad (2.16)$$

Performing differentiation in (2.9) and afterwards taking account of (2.13) and (2.15), we obtain

$$a^2 p^* (\rho, s) = P^* (s) p_1^* (\rho, s) \qquad (2.17)$$

$$p_1^* (\rho, s) = h_1^* (s) \left\{ \frac{\varphi_1^* (1, s)}{\sqrt{1-\rho^2}} - \int_\rho^1 \frac{\partial \varphi_1^* (x, s)}{\partial x} \frac{dx}{\sqrt{x^2-\rho^2}} \right\}$$

$$a^2 p (\rho, t) = L^{-1} [P^* (s) p_1^* (\rho, s)] \qquad (2.18)$$

It is possible to extract $P^* (s) = P = \text{const}$ from under the sign of the operator L^{-1} in the case of a constant force P.

3. Methods of calculation and results of a numerical solution of the contact problem in the case of a constant force. The fundamental required components of the problem of consolidation of a multilayered half-space under the impression of a circular press of constant force P are the time-dependent function of the press settlement $h_1 (t) = -2G_1 a w_1 (1, \rho, t) / P$ and the press pressure intensity $p_1 (\rho, t) = a^2 p (\rho, t) / P$ on the area $0 \leqslant \rho \leqslant 1$, which varies along the radius and depends on the time. A program in the language ALGOL is compiled to calculate the functions $h_1 (t)$ and $p_1 (\rho, t)$ in the general case of a multilayered half-space. Let us briefly examine the calculation methods used to realize numerical solutions of the contact problem.

In the general case, the function $\Delta_w^* (\beta, s)$ in (2.8) for the kernel of the integral equation (2.7) is expressed in terms of the solution of a high-order system (1.9). In the case of a homogeneous half-space, the function $\Delta_w^* (\beta, s)$ is expressed by a simple formula; for example, it has the following form for a completely permeable outer surface [1]

$$\Delta_w^* (\beta, s) = b (s + \beta b)^{-1} [(1 - \nu) \sqrt{s/c + \beta^2} - \nu\beta] - 1 + 2\nu$$
$$b = c\beta (1 - 2\nu) (1 - \nu)^{-2}$$

For a two-layered half-space with $N = 1$ the system (1.9) which contains nine algebraic equations for fixed β and s is solved effectively by the Gauss method with sampling of the principal element. Hence, from one minute to one hour of machine time is expended on a BESM-6 to calculate the functions $h_1 (t)$ and $p_1 (\rho, t)$ to 0.25% — 1% accuracy.

In the case $N > 1$ the following special method is used to accelerate the numerical solution of the system (1.9). We substitute the first three equations of the system (1.9) as the last $6N + 3$ equations and thereby convert this system into

$$RB = J \tag{3.1}$$

$$R = \left\| \begin{array}{c:c} A & D \\ \hdashline C & 0 \end{array} \right\| = \left\| \begin{array}{cccc:c} M_1R_2 & & & 0 & 0 \\ & M_2R_3 & & & \cdot \\ & & \ddots & & \cdot \\ 0 & & & M_N & P_{N+1} \\ \hdashline N_10 & \ldots & & 0 & 0 \end{array} \right\|, \quad J = \left\| \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{array} \right\| \tag{3.2}$$

The solution of (3. 1) is expressed by the formula $B = R^{-1}J$, where the inverse matrix R^{-1} for the block matrix R in (3. 2) has the form [13, 14]

$$R^{-1} = \left\| \begin{array}{c:c} S & G \\ \hdashline Q & F \end{array} \right\|, \quad G = \left\| \begin{array}{c} G_1 \\ G_2 \\ \vdots \\ G_N \end{array} \right\| \tag{3.3}$$

$$F = -(CA^{-1}D)^{-1}, \quad G = (A^{-1}D)F \tag{3.4}$$

$$A^{-1} = \left\| \begin{array}{ccccc} M_1^{-1} & M_1^{-1}R_2M_2^{-1} & \ldots & . & M_1^{-1}R_2M_2^{-1}\ldots R_NM_N^{-1} \\ 0 & M_2^{-1} & \ldots & . & M_2^{-1}R_3 \ldots R_NM_N^{-1} \\ . & . & \ldots & . & . \\ 0 & 0 & \ldots & M_{N-1}^{-1} & M_{N-1}^{-1}R_NM_N^{-1} \\ 0 & 0 & \ldots & 0 & M_N^{-1} \end{array} \right\| \tag{3.5}$$

and the matrices S and Q are not used any further.

According to the form of the free terms J in (3. 2), it is clear that it is sufficient to know only the third column from the right of the matrix R^{-1} in (3. 3) — (3. 5) which consists of the first columns of the block-matrices G and F , in order to determine the vector B (see (1. 9)). In calculating the required magnitudes of the contact problem it is sufficient to know the components of the vector B_1 in (1. 9), only on the outer surface $y = 1$ of the multilayered half-space, for whose calculation the first column of the matrix-square G_1 in the block-matrix G (3. 3) must be found, since

$$B_1(\beta, s) = G_1(\beta, s)I_1, \quad I_1 = \left\| \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right\| \tag{3.6}$$

Taling account of the form of the block-matrices C, D (3. 2) and A^{-1} (3. 5), we find the matrices needed to evaluate B_1 (β, s) (3. 6) by means of (3. 4)

$$F = -(N_1\Omega_NP_{N+1})^{-1}, \quad G_1 = (\Omega_NP_{N+1})F \tag{3.7}$$

$$\left(\Omega_N = M_1^{-1}\prod_{j=2}^{N} R_jM_j^{-1}\right)$$

The sixth order square matrices M_1^{-1} and $R_jM_j^{-1}$ are easily found in analytical form. Therefore, evaluation of the vector B_1 (β, s) (3. 6) reduces to calculation of the matrix Ω_N [6×6] and the first columns of the matrices F [3×3] and G_1 [6×3], by means of (3. 7), which is a fast-response and convenient algorithm for an electronic computer, and economical in its use of the operational memory. For comparison, let us note that evaluation of the vector B_1 (β, s) by using the Gauss method requires N^2 more operations and registers of the electronic computer memory.

For $h^*(s) \equiv 1$ the integral equation (2. 7) is converted to an integral equation on the segment $-1 \leqslant x \leqslant 1$ with the kernel

$$K(x-z, s) = \int_0^\infty \Delta_w^*(\beta, s) \cos[(x-z)\beta]\, d\beta \tag{3.8}$$

The converted integral equation is solved by replacing the integral by the tangent formula, whereupon a system of n-th order linear algebraic equations ($n = 100 \div 1000$) with Toeplitz matrices is obtained. The elements of this matrix are evaluated by the Filon method by means of (3.8). The solution of the algebraic system is obtained to the accuracy of order h^2 ($h = 2/(n+1)$) at the points $-1 + h/2, -1 + 3h/2, \ldots, 0, \ldots, 1 - 3h/2, 1 - h/2$ by a modified economical method [15]. After solving the system with a halved step, the solution of the integral equation is refined by the Runge method, after which the error of the solution is of order h^4. These questions are considered in greater detail in [16].

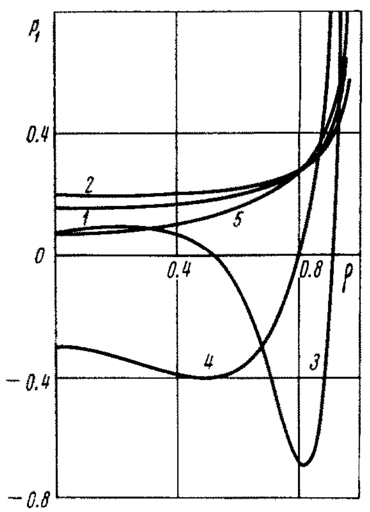

Fig. 2

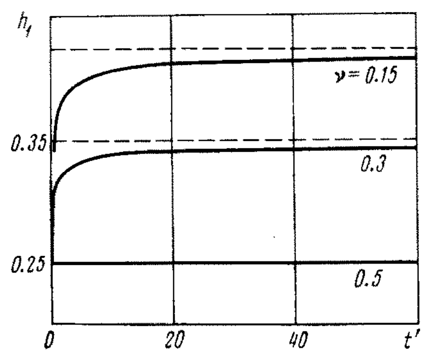

Fig. 3

Numerical values of $\varphi_1^*(x, s)$ obtained as a result of solving systems with the step h and $h/2$ are used to evaluate the integral (2.15) for $h_1^*(s)$ by the trapezoid formula. Then refinement by the Runge scheme is performed.

To calculate $p_1^*(\rho, s)$ by means of (2.17), the value of $\varphi_1^*(1, s)$ is obtained from the integral equation for $x = 1$. The integral from (2.17) is converted into

$$\int_0^{1-\rho} \frac{\psi(x, s)}{\sqrt{x}}\, dx, \quad \psi(x, s) = \frac{1}{\sqrt{x+2\rho}} \frac{\partial}{\partial x} \varphi_1^*(x+\rho, s) \tag{3.9}$$

The values of $\partial\varphi_1^*(x, s)/\partial x$ are calculated first to h^2 order accuracy by means of the values of $\varphi_1^*(x - h/2, s)$ and $\varphi_1^*(x + h/2, s)$, and then refined according to the Runge scheme. The integral (3.9) is evaluated by the Kotes formula for three points with the weight $1/\sqrt{x}$. The result is hence obtained to accuracy on the order of $h^2\sqrt{h}$.

An inverse Laplace-Carson transform is performed to obtain $h_1(t)$ (2.18) and $p_1(\rho, t)$ by using Lagrange polynomials [13, 17]. To do this it is necessary to know values of $h_1^*(s)$ (2.15) and $p_1^*(\rho, s)$ (2.17) at points of the real axis $s_0, 2s_0, \ldots, ms_0$ ($m = 3 \div 11$), where s_0 is selected so that the interval of time variation $t = T/s_0$ ($0.05 \leqslant T \leqslant 2.5$) would correspond to the time segment in which the solution must be obtained. The range of variation of T indicated in the parentheses has been selected from considerations of obtaining the contact problem solution to the accuracy needed in practice. Large errors originate for values of T outside this range. Finally, let us note that the calculations can be performed simultaneously for many values of the time t without a noticeable increase in machine time.

The solution of the problem of consolidation of a multilayered half-space under the effect of a press for $t = \infty$ agrees with the solution of the corresponding elasticity theory problem, and for $t = 0$ with the solution of the elasticity theory problem for $\nu_j = 0.5$ $(j = 1, 2, \ldots, N + 1)$ and previous values of G_j and χ_j and the other parameters [1]. In the case of a homogeneous half-space, the press pressure intensity $p_1(\rho, t)$ at $t = 0$ and $t = \infty$ is expressed by the general formula $p_1(\rho, t) = 1 / 2\pi \sqrt{1 - \rho^2}$. The functions $p_1(\rho, t)$ for a homogeneous half-space are represented in Fig. 2 for $t = 0$ and $t = \infty$ (curve *1*) and for $t' = 0.05$, $\nu = 0.15$ (curve *2*), while the time dependence of the settlement of the press $h_1(ct / a^2)$ is represented in Fig. 3 for $\nu = 0.15$, 0.3, 0.5. Also given in Fig. 2 are the functions $p_1(\rho, t)$ in the case of a two-layered half-space with the parameters $\nu_1 = \nu_2 = 0.3$, $\chi_1 = 100$ at $t = \infty$ for the values $\lambda = H_0 / a = 0.2, 0.5, 2$ (curves *3*, *4*, *5*, respectively). (The results differ slightly from those presented here for other values of the time $t > 0$).

REFERENCES

1. Pariiskii, B. S. and Nikishin, V. S., Axisymmetric Problem of Consolidating Multilayered Media. Computation Center, USSR Acad. of Science, Moscow, 1976.
2. Nikishin, V. S. and Shapiro, G. S., Three-dimensional Problems of Elasticity Theory for Multilayered Media. Computation Center, USSR Acad. of Science, Moscow, 1970.
3. Nikishin, V. S. and Shapiro, G. S., On contact problems for elastic multilayered media. Tr. Symp. on Mechanics of a Continuous Medium and Kindred Problems of Analysis, Tbilisi, 1971. Vol. 1, "Metsniereba", Tbilisi, 1973.
4. Biot, M. A., General theory of three-dimensional consolidation. J. Appl. Phys. Vol. 12, № 2, 1941.
5. Biot, M. A., General solutions of the equations of elasticity and consolidation for a porous material. J. Appl. Mech., Vol. 23, № 1, 1956.
6. McNamee, J. and Gibson, R. E., Displacement functions and linear transforms applied to diffusion through porous elastic media. Quart. J. Mech. and Appl. Math., Vol. 13, Pt. 1, 1960.
7. McNamee, J. and Gibson, R. E., Plane strain and axially symmetric problems of the consolidation of a semi-infinite clay stratum. Quart. J. Mech. and Appl. Math., Vol. 13, Pt. 2, 1960.
8. Szeer, G. and Gaszynski, J., Axisymmetric punch problem under condition of consolidation. Arch. Mech., Vol. 27, № 3, 1975.
9. Parton, V. S. and Kudriavtsev, B. A., Contact problem of the mechanics of deformation of porous viscoelastic media. Problems of the Mechanics of a Solid Deformed Body. "Sudostroenie", Leningrad, 1970.
10. Kerchman, V. I., Contact problem of the theory of consolidation of water-saturated medium. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, № 3, 1974.

11. Noble, B., Certain dual integral equations. J. Math. and Phys., Vol. 37, № 2, 1958.
12. Ufliand, Ia. S., Integral Transforms in Elasticity Theory Problems. "Nauka", Leningrad, 1967.
13. Lancosz, C., Applied Analysis. Prentice Hall, Englewood Cliffs, 1956.
14. Faddeev, D. K. and Faddeeva, V. N., Computational Methods of Linear Algebra. Fizmatgiz, Moscow-Leningrad, 1963.
15. Justice, J. H., An algorithm for inverting positive-definite Toeplitz matrices. SIAM, J. Appl. Math., Vol. 23, № 3, 1972.
16. Pariiskii, B. S., Economical Methods for the Numerical Solution of Equations in Convolutions and Systems of Algebraic Equations with Toeplitz Matrices. Computation Center, USSR Acad. of Sciences, Moscow, 1976.
17. Krylov, V. I. and Skoblia, N. S., Handbook on the Numerical Inversion of the Laplace Transforms. "Nauka i Tekhnika", Minsk, 1968.

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DIFFERENTIAL GAME OF GUIDANCE FOR A SYSTEM WITH SLACK AND INTEGRAL CONSTRAINTS

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The problem of guidance onto a convex target set of a system with slack is analyzed on the assumption that the realization of controls of the first player is hindered by integral constraints. Sufficient conditions of the problem solvability are formulated and an example is presented. This paper is related to [1 – 4].

1. Let us consider a controlled system described by the following vector differential equation:

$$\frac{dx}{dt} = A(t)x + C(v)u, \quad C(v) = \begin{Vmatrix} 0 & & 0 \\ 0 & & 0 \\ \vdots & \vdots & \vdots \\ \cos v & & -\sin v \\ \sin v & & \cos v \end{Vmatrix} \tag{1.1}$$

Here x is the n-dimensional phase vector of the system, u is two-dimensional control vector of the first player and v is the control of the second player. The realizations of the player controls are restricted by the conditions

$$\int_t^{\vartheta} \|u[\tau]\|^2 d\tau \leqslant \mu^2[t], \quad v[t] \in [-\alpha, +\alpha]$$

for any $t \in [t_0, \vartheta]$, $(\alpha < \pi/2)$.

The symbol $\|\cdot\|$ denotes the norm in the corresponding Euclidean space, $\mu[t]$ are the constraints imposed on the resources of the control of the first player, and ϑ denotes